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Normative positions within an algebraic approach to normative systems

Lars Lindahl [a], Jan Odelstad [b]

[a] *Faculty of Law, Lund University, Box 207, SE-221 00 Lund, Sweden*
[b] *Division of Computer Science, Gävle University, SE-801 76, Gävle, Sweden*

Abstract

The formal analysis of normative systems as initiated by Alchourrón and Bulygin can be complemented by the analysis of normative positions as pursued by Kanger, Lindahl, Sergot and Jones. The paper is a step towards integrating the two approaches within an algebraic theory of so-called Boolean quasi-orderings (*Bqo*'s). In the general *Bqo* theory presented, a number of theoretical tools are introduced and elucidated by theorems, in particular those of fragment, connection, coupling and pair coupling. Condition implication structures (*cis*'s) are models of the *Bqo* theory used for the representation of normative systems. A system of normative positions is introduced as a special kind of *cis*. The final section is devoted to an example exhibiting a legal mini-system where a *cis* of normative positions (*np-cis*) is joined to a descriptive *cis*.

Keywords: Normative systems; Alchourrón; Bulygin; Normative positions; Kanger; Implication structures; Boolean algebra; Deontic logic; Logic of action

1. Introduction

1.1. Normative systems

In their well-known book *Normative Systems* [1], Carlos E. Alchourrón and Eugenio Bulygin conceive of a normative system as a set of sentences deductively correlating pairs of sentences. According to them, a set α of sentences deductively correlates a pair $\langle p, q\rangle$ of sentences if q is a deductive consequence of $\{p\} \cup \alpha$, or, in symbols, if $q \in Cn(\{p\} \cup \alpha)$. For α to be a normative system the additional requirement is made that there be at least

E-mail addresses: lars.lindahl@jur.lu.se (L. Lindahl), jod@hig.se (J. Odelstad).

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one pair $\langle p, q\rangle$ where $q \in Cn(\{p\} \cup \alpha)$ such that p is a case and q is a solution. (A solution is a normative sentence expressed in terms of deontic operators for command, prohibition or permission.) As observed by Alchourrón and Bulygin, the statement $q \in Cn(\{p\} \cup \alpha)$ is equivalent to $(p \supset q) \in Cn(\alpha)$ where $\supset$ is the symbol for truth-functional implication.

If propositional logic is used as a basis it is usually presupposed that p, q are closed sentences with no free variables, i.e., for example, p is the sentence "Smith has promised to pay Jones \$100" and q is "Smith has an obligation to pay \$100 to Jones". In these sentences, individuals are referred to by individual constants (names). While it is true that a normative system may correlate sentences of this kind, a set of sentences containing individual names is not, however, an appropriate representation of a normative system. A normative system expresses general rules where no individual names occur. If the task is to represent a normative system this feature of generality has to be taken into account.

One way to do justice to the generality of norms is to represent a normative system in the language of predicate logic, i.e., by sentences like "For any x and y: if x has promised to pay \$$y$ to z then x has an obligation to pay \$$y$ to z." From such a general norm, instantiations concerning Smith, Jones and the amount of \$100 can be derived by predicate logic, i.e., if α is a set of general norms and p, q are closed sentences about Smith, Jones etc., sentences like the above $q \in Cn(\{p\} \cup \alpha)$ and $(p \supset q) \in Cn(\alpha)$ will come out true. Thus it will still be true that the normative system "normatively correlates" propositions about individuals.

When Alchourrón and Bulygin speak of normative "solutions" being correlated to "cases", however, they have in mind correlation of "generic" cases to "generic" solutions. They emphasize the distinction between individual and generic cases, and an analogous distinction holds for solutions. An individual case is a situation or a state of affairs. As such, appropriately, it should be described by a closed sentence. On the other hand, a generic case is a property or a set of individual cases, defined by a property.[1,2] Therefore, a "case" in the generic sense relevant to Alchourrón and Bulygin is an object described by an open sentence.[3] Consequently, when the expression $q \in Cn(\{p\} \cup \alpha)$ is said to express that α correlates q to p, q and p must be thought of as "open" sentences like "x has promised to pay \$$y$ to z", "x has an obligation to pay \$$y$ to z", not prefixed by any universal quantifier.[4]

[1] By an individual case is meant an element of the *UD* (the universe of discourse), where the *UD* is "a set of situations or states of affairs" [1, p. 28, and p. 10]. A generic case is described alternatively as a subset of the *UD* defined by a property, or as this defining property itself. The set of generic cases is called the *UC* (the universe of cases) [1, p. 29].

[2] Each property P is a bipartition of the set of individual cases, i.e., each either possesses or lacks property P. The set of properties in view for a legal problem is called the universe of properties, the *UP* [1, p. 10, pp. 26f, and p. 12].

[3] Since (generic) cases are described alternatively as sets, it might seem that another way to conceive of normative correlation would be in terms of set-theoretical inclusion between relations, where relations are understood to be sets of ordered n-tuples. This path (not suggested or commented upon in [1]) meets difficulties in the case that two relations R_1 and R_2 have different arity (for example, R_1 is unary and R_2 is binary). If R_1 and R_2 have different arity, $R_1 \cap R_2 = \emptyset$.

[4] Cf. [1, p. 49]: "According to the Deduction theorem the conditional sentence 'if x is 21, then he may administer his property' is a consequence of the set A if and only if the sentence 'x may administer his property'... is a consequence of the set composed of the set A ... and the sentence 'x is 21'... ".

Some features of the theory of Alchourrón and Bulygin can be summarized as follows. They conceive of a normative system α as represented by a set of universally quantified sentences where deontic operators occur in the consequents. They speak of cases and solutions as objects that are described by "open" sentences. The normative system correlates cases to solutions. The correlation is accomplished by there being a relation of consequence Cn such that α correlates solution q to case p to if $q \in Cn(\{p\} \cup \alpha)$.

A difficulty for the theory of Alchourrón and Bulygin is that, due to the features mentioned, the deductive machinery, expressed in terms of $p \supset q \in Cn(\alpha)$, is not based either on propositional logic or on predicate logic or on the logic of relations.[5] The question therefore arises whether it is possible to introduce a proper deductive machinery while retaining the idea that a normative system correlates objects called (generic) cases and solutions. (Instead of "cases" and "solutions" we might say "grounds" and "consequences".)

In a series of papers [10–13,17,18], the present authors have developed a theory of *condition implication structures* (*cis*'s) for dealing with the representation of normative systems. This theory was aimed specifically at analyzing the role of intermediate concepts (like contract, ownership) for coupling normative consequences to descriptive grounds within a normative system.

Our representation of normative systems in the papers referred to is similar to that of Alchourrón and Bulygin insofar as we study normative systems essentially as deductive mechanisms yielding outputs for inputs. (See, for example, [12, p. 91].) A difference, however, is that, in our approach, while input and output are particular, norms are explicitly general in character.

A logically satisfactory theory of implication between conditions requires a general logical theory as its basis. This theory is presented in the form of a theory of *Boolean quasi-orderings*, or *Bqo*'s for short. The *Bqo* theory is a general algebraic theory having models of many kinds. As will appear, the theory of *cis*'s is one of its models.

1.2. Normative positions

The theory of normative positions, in its modern logical form, essentially was shaped by the Swedish logician Stig Kanger [6,7]. Kanger's theory was inspired by the system of "fundamental jural relations" proposed by the American jurist W.N. Hohfeld in 1913, but took advantage of the development of formal logic. Of particular importance in this respect was Georg Henrik von Wright's reformulation of deontic logic. As realized by Kanger, however, standard deontic logic, with a deontic operator applied to sentences, is not adequate for expressing the Hohfeldian distinctions. The improvement proposed by Kanger was to combine a standard deontic operator *Shall* with an action operator *Do* (for "sees to it that") and to exploit the possibilities of external and internal negation of

[5] The character of the operator for implication is problematic. In connection with the consequence operation *Cn* the symbol $\supset$ is used. This symbol is said to be the symbol for material implication in truth-functional logic (see [1, p. 12, note 3]). Can an open sentence, like "x has promised to pay \$$y$ to z", be true or false? At p. 57, the implication "q implies Pp" (where P stands for "Permitted") is expressed by Pp/q, where a binary "dash" operator $/$ is used instead of $\supset$. Whether this is a significant change is not clear, since the dash operator $/$ is not specifically characterized or commented upon.

sentences where these operators are combined. Kanger's theory, expressed as a theory of types of rights, was further developed by Lars Lindahl in his three systems of types of normative positions [9]. Some further refinement of the systems have recently been made by Andrew J.I. Jones and Marek Sergot [4,5,19,20]. A special feature of the work of Jones and Sergot is that applications in computer science are in view.

To the Boolean connectives of negation, conjunction etc., Kanger added the modal expressions "Shall" and "Do". If F is a state of affairs and x is an agent,[6] Shall F is to be read "It shall be the case that F" and $\text{Do}(x, F)$ should be read "x sees to it that F". The expression $\text{May}F$ is an abbreviation for $\neg\text{Shall}\neg F$.

Kanger exploited the possibilities of combining the deontic operator Shall with the action operator Do. One example is Shall $\text{Do}(x, F)$ which means that it shall be that x sees to it that F; another is $\neg$Shall $\text{Do}(y, \neg F)$ which means that it is not the case that it shall be that y sees to it that not F.

The logical postulates for Shall and Do assumed by Kanger are as follows (where $\longrightarrow$ is a relation of logical consequence[7]):

1. If $F \longrightarrow G$, then Shall $F \longrightarrow$ Shall G.
2. (Shall F & Shall G) $\longrightarrow \text{Shall}(F \& G)$.
3. Shall $F \longrightarrow \neg$Shall $\neg F$.[8]
4. If $F \longrightarrow G$ and $G \longrightarrow F$, then $\text{Do}(x, F) \longrightarrow \text{Do}(x, G)$.
5. $\text{Do}(x, F) \longrightarrow F$.

The construction of types of normative position will be described in a later section (Section 4.1.1). One example is $\text{Shall}\,\text{Do}(x, F)$, expressing that it shall be that x sees to it that F, another is $\text{Shall}(\neg\text{Do}(x, F) \,\&\, \neg\text{Do}(x, \neg F))$, expressing that it shall be that x is passive with regard to F.[9]

The systems of normative positions can serve as a tools for describing the normative positions of different agents $x, y, z, \ldots$ with regard to states of affairs $F, G, H, \ldots$.[10] A set

[6] A state of affairs in Kanger's sense might be, for example, that Mr. Smith gets back the money lent by him to Mr. Black, or that Mr. Smith walks outside Mr. Black's shop.

[7] The principles assumed by Kanger for the relation of logical consequence ($\longrightarrow$) are as follows:

(i) If F and $F \longrightarrow G$, then G;
(ii) If $F \longrightarrow G$, then $\neg G \longrightarrow \neg F$;
(iii) If $F \longrightarrow G$ and $G \longrightarrow H$, then $F \longrightarrow H$.

See [8, p. 88, note 3].

[8] Expressed in terms of May, the postulates become:

1. If $F \to G$, then May $F \to$ May G.
2. $\text{May}(F \vee G) \to \text{May}F \vee \text{May}G$.
3. $\neg\text{May}F \to \text{May}\neg F$.

[9] In [4] and [19,20], an alternative scheme is used, which, for one-agent types yields the same results as the one used here and in [9, Chapter 3]), provided that the deontic postulates 1–3 above are assumed, but which does not depend on these specific postulates.

[10] For example, if x is the Swedish Government and F is the state of affairs that a paper on normative positions by Lindahl is published in Sweden, the position, according to Swedish law, of x with regard to F can be described by $\text{Shall}(\neg\text{Do}(x, F) \,\&\, \neg\text{Do}(x, \neg F))$, expressing that the Government is not allowed either to bring about or prevent the publication.

of such descriptions, however, is not a representation of a normative system. This is due to the fact that a normative system is not a description of the actual normative positions of individuals. Rather the essential feature of a normative system consists in so-called normative correlations, i.e., as jurists might say, in correlating normative consequences to operative facts.

In the present paper, the *Bqo* approach to normative systems will be combined with the Kanger–Lindahl theory of normative positions.

2. Boolean quasi-orderings

2.1. Basic notions

Though the theory of Boolean quasi-orderings (*Bqo's*) will be used here for special purposes, the notion of a Boolean quasi-ordering is of a general character. A *Bqo* is any relational structure based on a Boolean algebra and satisfying certain requirements.[11]

As is well-known, the postulates for a Boolean algebra $\langle B, \wedge, \vee, '\rangle$, with $\top$ as the unit element and $\bot$ as the zero element, are as follows:

$$\begin{array}{ll}
a \wedge b = b \wedge a & a \vee b = b \vee a \\
a \wedge (b \wedge c) = (a \wedge b) \wedge c & a \vee (b \vee c) = (a \vee b) \vee c \\
a \wedge (a \vee b) = a & a \vee (a \wedge b) = a \\
a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) & a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \\
a \wedge \top = a & a \vee \bot = a \\
a \wedge a' = \bot & a \vee a' = \top
\end{array}$$

Since $\vee$ is definable by $a \vee b = (a' \wedge b')'$, in what follows, instead of $\langle B, \wedge, \vee, '\rangle$ we use the notation $\langle B, \wedge, '\rangle$ for a Boolean algebra with B as its domain.

Definition 1. A Boolean quasi-ordering (*Bqo*) is defined as follows. Assume that $\langle B, \wedge, '\rangle$ is Boolean algebra, and that R is a binary, reflexive and transitive, relation on B. Then, the relational structure $\langle B, \wedge, ', R\rangle$ is a `Boolean quasi-ordering` if R satisfies the following conditions for all a, b and c in B:

(1) aRb and aRc implies $aR(b \wedge c)$.
(2) aRb implies $b'Ra'$.
(3) $(a \wedge b)Ra$.
(4) not $\top R \bot$.

A Boolean quasi-ordering $\langle B, \wedge, ', R\rangle$ will be denoted by $\mathcal{B}$ (calligraphic). Furthermore, Boolean quasi-orderings $\langle B, \wedge, ', R\rangle$ and $\langle B_i, \wedge, ', R_i\rangle$ will be denoted by $\mathcal{B}$ and $\mathcal{B}_i$. The

[11] The theory of Boolean quasi-orderings was introduced in [17] and further developed in [13]. In two earlier papers [10,11] a theory with the same aim but based on lattice theory was presented. In [12] a study was made of systems weaker than Boolean quasi-orderings but generating such orderings.

indifference part of R is denoted Q and is defined by: aQb if and only if aRb and bRa. Similarly, the strict part of R is denoted S and is defined by: aSb if and only if aRb and not bRa.

Let $\leqslant$ be the partial ordering determined by the Boolean algebra $\langle B, \wedge, ' \rangle$.[12] From requirement (3) for *Bqo*'s it follows that $a \leqslant b$ implies aRb.[13] As a special case, we note that $a \leqslant \bot$ implies $aR\bot$, and $\top \leqslant a$ implies $\top Ra$.

Next we introduce some general notions relating to *Bqo*'s. If $\langle B, \wedge, ', R \rangle$ is a *Bqo* then we say that the Boolean algebra $\langle B, \wedge, ' \rangle$ is the *reduct* of $\langle B, \wedge, ', R \rangle$. In what follows, the reduct $\langle B, \wedge, ' \rangle$ of a *Bqo* $\mathcal{B}$ will be denoted $\mathcal{B}^{red}$.

Suppose that $\mathcal{B} = \langle B, \wedge, ', R \rangle$ is a *Bqo* and Q is the indifference part of R. The *quotient algebra* of $\mathcal{B}$ with respect to Q is a structure $\langle B/Q, \cap, -, \preceq_Q \rangle$ such that $\langle B/Q, \cap, - \rangle$ is a Boolean algebra and $\preceq_Q$ is the partial ordering determined by this algebra.[14] The natural mapping of $\langle B, \wedge, ' \rangle$ onto $\langle B/Q, \cap, - \rangle$ is a homomorphism (see [18]). We call $\langle B/Q, \cap, - \rangle$ the *quotient reduction* of $\mathcal{B}$. Thus there are two Boolean algebras which should be kept apart, namely $\mathcal{B}^{red}$, i.e., the reduct of $\mathcal{B}$, and the quotient reduction of $\mathcal{B}$. If the quotient reduction of $\mathcal{B}$ is isomorphic to $\mathcal{B}^{red}$, $R = \leqslant$, and we say that $\mathcal{B}$ is *conservatively reducible*.

Although, by a transition to equivalence classes, from a Boolean quasi-ordering we get a new Boolean algebra, in what follows we will not make this transition. The point is that, in the models we have in mind, where the domain of a *Bqo* is a set of conditions, even though, for two conditions a and b, it holds that aQb (and therefore a and b belong to the same Q-equivalence class), we may want to distinguish a and b because they may have different meaning. Therefore, there is a point in remaining within the framework of Boolean quasi-orderings as defined above.[15]

2.1.1. Atoms of a Bqo

The notion of an atom in a Boolean algebra is standard, but we also introduce the notion of a dual atom.

Definition 2. Let $\langle B, \wedge, ' \rangle$ be a Boolean algebra. Then

(1) a is a an `atom` in $\langle B, \wedge, ' \rangle$ if (i) not $a = \bot$ and (ii) $b \leqslant a$ implies that either $b = a$ or $b = \bot$, i.e., a is an atom in $\langle B, \wedge, ' \rangle$ if there is no $b \in B$ such that $\bot < b$ and $b < a$.

(2) a is a `dual atom` in $\langle B, \wedge, ' \rangle$ if (i) not $a = \top$ and (ii) $a \leqslant b$ implies that either $b = a$ or $b = \top$, i.e., a is a dual atom in $\langle B, \wedge, ' \rangle$ if there is no $b \in B$ such that $a < b$ and $b < \top$.

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[12] As usual, $\leqslant$ is defined by $a \leqslant b$ if and only if $a \wedge b = a$.

[13] By definition, $a \leqslant b$ implies $a \wedge b = a$ and, since, by (3), it holds that $a \wedge bRb$, it follows that aRb.

[14] Note that the slash sign / will be used in two ways. Firstly, it is used for denoting equivalence classes like in B/Q. Then B is presupposed to be a set and Q an equivalence relation. Secondly it used to denote the restriction of a relation to a particular subset of its field, like in R/B_1 (where R is a relation over B and B_1 is a subset of B). When / is used in this way, R is presupposed to be a relation and B_1 a set.

[15] The theory of Boolean quasi-orderings is of a very general character, and it is easy to see that we can construct a model of this theory out of a first order theory Σ. Consider the structure $\langle B, \wedge, ', R \rangle$ where $\langle B, \wedge, ' \rangle$ is the Lindenbaum algebra of the predicate calculus. Let R be the quasi-ordering on B determined by the Lindenbaum algebra of Σ. Then $\langle B, \wedge, ', R \rangle$ is a *Bqo*. Cf. [2, p. 61], and [3, p. 73].

From Lemma 4 below it follows that a is an atom in the Boolean algebra $\langle B, \wedge, ' \rangle$ iff a' is a dual atom in $\langle B, \wedge, ' \rangle$.

As usual, a Boolean algebra $\langle B, \wedge, ' \rangle$ is said to be *atomic* if for each $b \in B$ such that not $b \leqslant \bot$ there is an atom a of $\langle B, \wedge, ' \rangle$ such that $a \leqslant b$.

The notions of atom and dual atom are extended to *Bqo*'s.

Definition 3. Let a be an element of a *Bqo* $\mathcal{B}$. Then,

(1) a is an `atom` in $\mathcal{B}$ if (i) not $aQ\bot$ and (ii) bRa implies that either $bQ\bot$ or bQa, i.e. a is an atom if there is no $c \in B$ such that $\bot Sc$ and cSa,

(2) a is a `dual atom` in $\mathcal{B}$ if (i) not $aQ\top$ and (ii) aRb implies that either $bQ\top$ or bQa, i.e. a is a dual atom if there is no $b \in B$ such that aSb and $bS\top$.[16]

Also we say that a *Bqo* $\mathcal{B}$ is *atomic* if for each $b \in B$ such that not $bR\bot$ there is an atom a of $\mathcal{B}$ such that aRb.

In what follows, $at(\mathcal{B})$, $dual\ at(\mathcal{B})$, $at(\mathcal{B}^{red})$, $dual\ at(\mathcal{B}^{red})$ denote the set of atoms and dual atoms, respectively, of $\mathcal{B}$, $\mathcal{B}^{red}$. Furthermore, for $a \in B$, $at(a)$ denotes the set of all $b \in at(\mathcal{B}^{red})$ such that $b \leqslant a$. Note that, if $\mathcal{B}^{red}$ is atomic, then, for $a \in B$ it holds that $a = \sup_{\leqslant} at(a)$ (where $\sup_{\leqslant}$ is supremum with respect to $\leqslant$, as defined in the usual way for partial orderings).

Lemma 4. *Let a be an element of a* Bqo *$\mathcal{B}$. Then, a is an atom in a Bqo $\mathcal{B}$ if and only if a' is a dual atom in $\mathcal{B}$.*

Proof. Suppose that a is an atom in $\mathcal{B}$. Hence, not $aQ\bot$ and from this follows not $a'Q\top$. Now suppose that $a'Rb$, which implies that $b'Ra$. Since a is an atom it follows that $b'Q\bot$ or $b'Qa$, which is equivalent to $bQ\top$ or bQa'. Thus, a' satisfies the requirements for being a dual atom. The other part of the equivalence in the lemma is proved analogously. □

The concept of an atom in a *Bqo* is different from the concept of an atom in a Boolean algebra. It is important to distinguish between the two kinds of atoms, since it can be the case that a is an atom in $\mathcal{B}^{red}$, while a is not an atom in $\mathcal{B}$, and conversely.[17] For the interrelation between the two kinds of atoms, however, the following holds.

Lemma 5. *Let $\mathcal{B}$ be a Boolean quasi-ordering. Then,*

(1) *if a is an atom in $\mathcal{B}^{red}$, then a is an atom in $\mathcal{B}$ or $aQ\bot$,*

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[16] With regard to a Boolean algebra $\langle B, \wedge, ' \rangle$, the notion of dual atom can be explained as follows. If ϕ is a statement on Boolean algebras, its *dual statement* $d\phi$ is obtained by systematically exchanging the symbols $\vee, \wedge, \bot, \top$ in ϕ. If ϕ is the statement that a is an atom in $\langle B, \wedge, ' \rangle$ the dual statement in this sense is that a is a dual atom in $\langle B, \wedge, ' \rangle$, meaning that (i) not $\top = a$ and (ii) $a \leqslant b$ implies that either $\top = b$ or $b = a$. Likewise as the notion of atom is extended to *Bqo*'s by substituting R for $\leqslant$ and Q for $=$, the notion of dual atom is extended to *Bqo*'s in the same way.

[17] Suppose $a \wedge b$ is an atom in $\mathcal{B}$ as well as in its reduct. Then, if $a \wedge b\ Q\ a$ while $a \wedge b < a$, a is an atom in $\mathcal{B}$ but not in the reduct of $\mathcal{B}$. Suppose a is an atom in the reduct of $\mathcal{B}$, while $aQ\bot$. Then a is an atom in the reduct of $\mathcal{B}$ but a is not an atom in $\mathcal{B}$.

(2) *if a is a dual atom in $\mathcal{B}^{red}$ then a is a dual atom in $\mathcal{B}$ or $\top Ra$.*

Proof. (1) Suppose that, for some $b \in B$, it holds that not aRb. It follows that not $a \leqslant b$, hence, since a is an atom in $\mathcal{B}^{red}$, $a \leqslant b'$, i.e., $b \leqslant a'$, and so, bRa'.[18] Consequently, if bRa and not aRb, then $bRa \wedge a'$, i.e., $bR\bot$. Thus there is no $b \in B$ such that bRa and not aRb and not $bR\bot$. Therefore, according to Definition 3, if not $aR\bot$, a is an atom in $\mathcal{B}$.

(2) is proved analogously. □

Also note that if a is an atom in $\mathcal{B}$ and aQb, then b is an atom in $\mathcal{B}$. Furthermore, if a and b are atoms in $\mathcal{B}$, then $a \wedge bQ\bot$ or aQb.

Lemma 6. *Suppose that the Boolean quasi-ordering $\mathcal{B}$ is finite. Then if a is an atom in $\mathcal{B}$, there is an atom b in $\mathcal{B}^{red}$ such that aQb.*

Proof. Suppose that a is an atom in $\mathcal{B}$ but that a is not an atom in $\mathcal{B}^{red}$. Since $\mathcal{B}^{red}$ is finite and hence atomic, $a = \sup_{\leqslant} at(a)$. By Lemma 5, if $b \in at(a)$ then b is an atom in $\mathcal{B}$ or $bQ\bot$, and further, $b \leqslant a$ which implies bRa. Suppose not $bQ\bot$. Then, if bSa, a is not an atom in $\mathcal{B}$, which contradicts the assumption, hence bQa. Thus we have proved that if $b \in at(a)$ then $bQ\bot$ or bQa. Suppose now that for all $b \in at(a)$, $bQ\bot$. Since $\mathcal{B}$ is finite, $\sup_{\leqslant} at(a)Q\bot$, i.e., $aQ\bot$, which implies a contradiction. Thus there is $b \in at(a)$ such that not $bQ\bot$ and bQa. □

2.1.2. Least upper bound and greatest lower bound in a Bqo

The notions of least upper bound and greatest lower bound are usually defined for partial orderings not for quasi-orderings. Since the relation R is a quasi-ordering, we introduce the following definitions of these notions for quasi-orderings.

Definition 7. Let R be a quasi-ordering of a set A with $X \subseteq A$, and let $a \in A$. Then, with respect to R,

- a is an upper bound for X, denoted $a \in \mathfrak{ub}_R X$, if xRa for all $x \in X$,
- a is a lower bound for X, denoted $a \in \mathfrak{lb}_R X$, if aRx for all $x \in X$,
- a is a least upper bound for X, denoted $a \in \mathfrak{lub}_R X$, if a is an upper bound for X and aRb for all upper bounds b for X,
- a is a greatest lower bound for X, denoted $a \in \mathfrak{glb}_R X$, if a is a lower bound for X and bRa for all lower bounds b for X.

We note that (in contrast to what holds for partial orderings) a least upper bound or a greatest lower bound relative to a quasi-ordering $\langle A, R\rangle$ need not be unique. Hence $\mathfrak{lub}_R X$ and $\mathfrak{glb}_R X$ denote subsets of X, not elements. However, if $x, y \in \mathfrak{lub}_R X$ or $x, y \in \mathfrak{glb}_R X$

[18] The proof that if a is an atom in a Boolean algebra $\langle B, \wedge, '\rangle$ and $b \in B$, then not $(a \leqslant b)$ implies $a \leqslant b'$, is standard. We have $a \wedge b \leqslant a$, hence, by the definition of atom, if a is an atom in $\langle B, \wedge, '\rangle$, $a \wedge b = a$ or $a \wedge b = \bot$, i.e., $a \leqslant b$ or $a \leqslant b'$. Both cannot hold, since then we get $a = \bot$, which contradicts the assumption that a is an atom. We note that the proof does not presuppose that $\langle B, \wedge, '\rangle$ is finite or atomic.

then xQy. Furthermore, if $x \in \mathfrak{lub}_R X$ and xQy, then $y \in \mathfrak{lub}_R X$, and analogously for $\mathfrak{glb}_R X$.

3. Cis models of the Bqo theory

By a condition implication structure (*cis*) is meant a *Bqo* $\mathcal{B} = \langle B, \wedge, ', R\rangle$ such that B is a domain of *conditions*, and R is such that aRb represents that a *implies* b. This reading is justified since, if a and b are ν-ary conditions, aRb is the representation of

$$\forall x_1, \ldots, x_\nu : a(x_1, \ldots, x_\nu) \to b(x_1, \ldots, x_\nu).$$

If $\mathcal{S}$ is a normative system represented by $\mathcal{B}$, a normative correlation in $\mathcal{S}$ is represented by $a_1 R a_2$, where $a_1, a_2 \in B$, and a_1 is descriptive while a_2 is normative.[19]

In simple cases, conditions can be denoted by expressions, using the sign of the infinitive, such as “to be 21 years old”, “to be a citizen of the US”, “to be a child of”, “to be entitled to inherit”, or by corresponding expressions in the ing-form, like “being 21 years old” etc. Often, however, conditions should appropriately be expressed by open sentences, like “x’s promises to pay \$$y$ to z”, “x is a citizen of state y”, “x is entitled to inherit y”.

When a condition is expressed by an open sentence, free variables like $x, y, z, \ldots.$ occurring in the sentence merely are place-holders for expressing the condition in a convenient way and keeping track of the order of the places. In simple cases like, “committing murder implies being liable to imprisonment”, place-holders are not needed.

A condition like “x promises to pay \$$y$ to z” is said to be *fulfilled* or *non-fulfilled* by a particular triple, such as ⟨Smith, 100, Jones⟩. The fulfillment of a condition by a particular n-tuple of individuals is a state of affairs, and is expressed by a sentence naming the individuals of the n-tuple.

A framework with implication between conditions seems to accord with the presupposed ontology of legal language where terms such as “citizenship”, “inheritance”, “ownership”, denote conditions that are treated as objects between which there is an implicative relation of “ground-consequence”, often expressed in terms of “gives rise to” or “causes”, or “implies”. Thus inheritance is said to give rise to ownership and ownership is said to imply a bundle of liberties, claims, and immunities.

Conditions have many affinities with relations, if, as is usual, relations are regarded extensionally as sets of ordered n-tuples. Obviously, the operations of negation, conjunction and disjunction for conditions have as counterparts the operations of complement, intersection and union for relations. However, if R_1 and R_2 are relations of different arity, their intersection $R_1 \cap R_2$ is empty and their union is not a relation. For example the intersection between a set of pairs and a set of triples is empty, and the union of a set of pairs and a set of triples is not a relation. The case is different with conditions, where sameness of arity is not presupposed for intersection and union.

[19] The present section on condition implication structures recapitulates ideas presented in earlier papers. See, in particular, [13,17].

A full algebraic treatment of conditions, having the same expressive power as predicate logic, presupposes a framework of cylindric algebra.[20] This extensive framework will not be provided in what follows. Rather, in what follows, only a fragment of a theory of conditions will be developed. The treatment is made algebraic within the framework of the *Bqo* theory, where relations between conditions can be introduced.

3.1. The arity of conditions

Conditions can be of different arity (unary, binary etc.). Examples of (binary) conditions are: to be the father of, to be the guardian of, to administer the property of, having the obligation to administer the property of, having the right to compensation for, etc. Where a condition is represented by an expression $a(x_1, \ldots, x_\nu)$ we presuppose that $x_1, \ldots, x_\nu$ are free variables which function as place-holders, and that $a(x_1, \ldots, x_\nu)$ is a sentence-form. If a and b are ν-ary conditions, we form compound ν-ary conditions by $'$ (negation), $\wedge$ (conjunction), and $\vee$ (disjunction). $\top$ is the ν-ary empty condition such that for no $x_1, \ldots, x_\nu$, $\bot(x_1, \ldots, x_\nu)$, and $\bot$ is the ν-ary universal condition such that for all $x_1, \ldots, x_\nu$, $\top(x_1, \ldots, x_\nu)$.

The arity of a' is always the same as the arity of a. For example, since being a woman is a unary condition, not being a woman is unary as well. For conjunction the following rule is adopted. If a is μ-ary and b is ν-ary and $\phi = \max\{\mu, \nu\}$, then, for all $x_1, \ldots, x_\phi$, $(a \wedge b)(x_1, \ldots, x_\phi)$ iff $a(x_1, \ldots, x_\mu)$ and $b(x_1, \ldots, x_\nu)$. Thus the arity of $a \wedge b$ equals the greatest of the arities of a and b. For example, if a is the condition to be a woman and b is the condition to be a parent of, then $a \wedge b$ is the condition of being a mother of. As regards the identity relation $=$ for conditions, if a is μ-ary and b is ν-ary and $\phi = \max\{\mu, \nu\}$, $a = b$ implies that, for all $x_1, \ldots, x_\phi$, $a(x_1, \ldots, x_\mu)$ iff $b(x_1, \ldots, x_\nu)$.

3.2. The implicative relation R

Since any *cis* is a model of the *Bqo* theory, R is reflexive, transitive and fulfills the requirements (1)–(4) above for *Bqo*'s, i.e.,

(1) aRb and aRc implies $aR(b \wedge c)$.
(2) aRb implies $b'Ra'$.
(3) $(a \wedge b)Ra$.
(4) not $\top R \bot$.

While the conditions in the domain of a *cis* are general (like x's promising to pay \$$y$ to z, where x, y, z are place-holders, not individual constants), a *cis* can be applied to individuals. This is accomplished by deductions of the following kind, where i, j are individual constants, for instance names of individuals. Let $\mathcal{B} = \langle B, \wedge, ', R\rangle$ be a *cis*, let a, b be ν-ary

[20] In choosing our approach of considering Boolean algebras of conditions, we have been inspired by some lectures of Stig Kanger's, given in the Fall of 1977. In these lectures, Kanger started developing an algebraic theory of conditions, based on Boolean and cylindric algebras.

conditions that are elements of B, and let $j_1, \ldots, j_\nu$ be names of individuals. From an instantiation of a, using the relation R of the *cis* $\mathcal{B}$, we can derive an instantiation of b by the scheme of inference:

$$\begin{array}{l} a(j_1, \ldots, j_\nu) \\ aRb \\ \vdots \\ b(j_1, \ldots, j_\nu). \end{array}$$[21]

If aRb holds in a *cis* $\mathcal{B} = \langle B, \wedge, ', R\rangle$, we say that an instantiated condition $a(j_1, \ldots, j_\nu)$ implies another instantiated condition $b(j_1, \ldots, j_\nu)$ *by way of* $\mathcal{B}$.

In the approach to normative systems adopted in this paper, a normative system is represented by a *cis* and the *cis* is seen as a deductive mechanism for inferring normative consequences from descriptive grounds. (Cf. Section 1.1.) In the *cis* representation, the norms belonging to such a system are represented by a relation R on the domain of conditions. Thus general rules in a normative system $\mathcal{S}$ are represented by an implicative relation R between conditions, i.e., the set of rules is a set of ordered pairs of conditions. For example, if a is the condition of x's promising to pay \$$y$ to z, and b is the condition of x's having an obligation to pay \$$y$ to z, a rule in $\mathcal{S}$ correlating b to a is represented by aRb.

If

(i) $\mathcal{B} = \langle B, \wedge, ', R\rangle$ is a *cis*,
(ii) aRb holds in $\mathcal{B}$, and
(iii) $\mathcal{B}$ represents a normative system $\mathcal{S}$,

then we say that it follows from $\mathcal{S}$ that a implies b, or that, according to $\mathcal{S}$, a implies b, or that, according to $\mathcal{S}$, a is a ground for b (and b is a consequence of a).[22] Also, we will say that the instantiation $b(j_1, \ldots, j_\nu)$ is deducible from the instantiation $a(j_1, \ldots, j_\nu)$ according to $\mathcal{S}$ if aRb holds in a *cis* $\mathcal{B} = \langle B, \wedge, ', R\rangle$ representing $\mathcal{S}$. Thus, a normative system can be seen as a system of general norms, serving as a deductive mechanism for inferring instantiated conditions from instantiated conditions.[23]

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[21] Since, in a *cis* $\langle B, \wedge, ', R\rangle$, aRb is the representation of

$$\forall x_1, \ldots, x_\nu : a(x_1, \ldots, x_\nu) \to b(x_1, \ldots, x_\nu),$$

the scheme of inference is obviously valid.

[22] If $\mathcal{S}$ is a normative system, the statement "according to $\mathcal{S}$, a implies b" is often said to express a normative proposition (cf. [1, p. 121]). We note that this normative proposition does not follow from aRb alone but from the conjunction of (i)–(iii).

[23] The approach of [15,16] is similar insofar as a normative system is seen by them as a deductive mechanism for inferring an output of Boolean propositions from an input of Boolean propositions. Their framework for representing normative systems, however, is different from ours in several respects.

4. Deontic cis models

4.1. The cis version of normative positions

4.1.1. The Kanger–Lindahl theory

In earlier papers based on the *Bqo* approach, the present authors did not deal with the fine-grained structure of a normative *cis*. A natural approach is to formulate this structure in terms of deontic logic and action logic. As stated in the introduction, therefore, one aim of present paper is to combine the *Bqo* approach to normative systems with an explicitly algebraic version of the Kanger–Lindahl theory of normative positions. (See [9, Chapters 3–5]). The *Bqo* and *cis* approach of the present paper, aiming at the analysis of normative correlations, is based on Boolean algebras of conditions. Therefore, the theory formulated by Kanger and Lindahl in terms of modal operators Shall and Do is reformulated in a *cis* model of the *Bqo* theory.

Since the present paper is a first step towards integrating normative positions within the *cis* representations of normative systems, the types of normative positions dealt with are chosen so as to be of a relatively simple kind.

The system of one-agent types of normative position, in the sense of [9, Chapter 3] are based on the logic of Do and Shall as stated above in Section 1.2. The types are constructed in the following way. If F is a state of affairs, let $\pm F$ stand for either of F or $\neg F$. Then first, a list of three one-agent types of action is obtained as follows.

Starting from the scheme $\pm\,\mathrm{Do}(x, \pm F)$, a list is made of all maximal and consistent conjunctions such that each conjunct satisfies this scheme. Consistency is consistency according to the logic of Do, and maximality means that if we add any further conjunct satisfying the scheme, then this new conjunct either is inconsistent with the original conjunction or redundant.[24] By this procedure a list of three conjunctions is obtained, which are denoted $A_1(x, F)$, $A_2(x, F)$, $A_3(x, F)$:

$$A_1(x, F)\colon\ \mathrm{Do}(x, F),$$

$$A_2(x, F)\colon\ \mathrm{Do}(x, \neg F),$$

$$A_3(x, F)\colon\ \neg\,\mathrm{Do}(x, F)\ \&\ \neg\,\mathrm{Do}(x, \neg F).$$

Given the underlying logic of Do, the one-agent formulas are mutually incompatible and their disjunction is a logical truth. (We note that the formula $\mathrm{Do}(x, F)\ \&\ \mathrm{Do}(x, \neg F)$ is inconsistent in the logic of Do, since it implies $F\ \&\ \neg F$.)

The formula $A_3(x, F)$, i.e., $\neg\,\mathrm{Do}(x, F)\ \&\ \neg\,\mathrm{Do}(x, \neg F)$, expresses x's passivity with regard to F. This formula therefore, in what follows, is expressed by $\mathrm{Pass}(x, F)$.

The three one-agent formulas are said to express *action*, *passivity* and *counter-action* with regard to F, as they state that x sees to it that F, that x is passive with regard to F, and that x sees to it that not F, respectively.

[24] There are several equivalent descriptions of how types of normative positions are constructed. The device of $\pm$-schemes and "maxi-conjunctions" for obtaining a short and convenient description was invented by David Makinson. See [14].

Next starting from the scheme $\pm$ May $A_i(x, F)$ $(1 \leqslant i \leqslant 3)$, a list is made of all maximal and consistent conjunctions, such that each conjunct satisfies this scheme. (Consistency means consistency according to the logic of Shall and May.) By this procedure a list of seven conjunctions is obtained, which express one-agent normative positions with regard to F, and are denoted $T_1(x, F), \ldots, T_7(x, F)$:[25]

$T_1(x, F)$: MayDo(x, F) & MayPass(x, F) & MayDo$(x, \neg F)$.

$T_2(x, F)$: MayDo(x, F) & MayPass(x, F) & $\neg$ MayDo$(x, \neg F)$.

$T_3(x, F)$: MayDo(x, F) & $\neg$ MayPass(x, F) & MayDo$(x, \neg F)$.

$T_4(x, F)$: $\neg$ MayDo(x, F) & MayPass(x, F) & MayDo$(x, \neg F)$.

$T_5(x, F)$: MayDo(x, F) & $\neg$ MayPass(x, F) & $\neg$ MayDo$(x, \neg F)$.

$T_6(x, F)$: $\neg$ MayDo(x, F) & MayPass(x, F) & $\neg$ MayDo$(x, \neg F)$.

$T_7(x, F)$: $\neg$ MayDo(x, F) & $\neg$ MayPass(x, F) & MayDo$(x, \neg F)$.[26]

Given the underlying logic of Do and May, the one-agent formulas are mutually inconsistent and their disjunction is a logical truth. (We note that, since $A_1(x, F) \vee A_2(x, F) \vee A_3(x, F)$ is a logical truth, formula

$\neg$ MayDo(x, F) & $\neg$ MayPass(x, F) & $\neg$ MayDo$(x, \neg F)$

is logically false, according to the logic of Shall and May.)

T_i is said to be the *converse* of T_j if it holds that $T_i(x, F)$ if and only if $T_j(x, \neg F)$, and T_i is *neutral* if it is its own converse (cf. [9, p. 92]). In this sense T_2 is the converse of T_4 and T_5 the converse of T_7, while T_1, T_3, T_6 are neutral.

4.1.2. Normative position cis

The simplest way to combine the *Bqo* approach to normative systems with the theory of one-agent normative positions is to transform the one-agent formulas $T_1(x, F), \ldots, T_7(x, F)$ into seven conditions. By such a transformation the theory of one-agent normative positions can be expressed in algebraic version within the *Bqo* framework.

Suppose q is a ν-ary condition. Then $T_i q$ (with $1 \leqslant i \leqslant 7$) is the $\nu + 1$-ary condition such that

$$T_i q(y_1, \ldots, y_\nu, x) \quad \text{iff} \quad T_i\big(x, q(y_1, \ldots, y_\nu)\big),$$

where $T_i(x, q(y_1, \ldots, y_\nu))$ is the i-th formula of one-agent normative positions. Thus T_i, when occurring in $T_i q$, is an operator on conditions, defined in terms of one-agent type T_i. A set $\{T_1 q, \ldots, T_7 q\}$ of seven conditions is obtained, and Boolean compounds of these seven conditions are formed by $\wedge, ', \vee$. Note that for example, $(T_1 q \vee T_2 q)(y_1, \ldots, y_\nu, x)$

[25] See [9, p. 92].

[26] The numbering of the T_i conforms to the numbering of the corresponding one-agent types of normative position in [9]. The numbering suits the representation of the types in a Hasse diagram, exhibiting how the types are partially ordered by the relation "less free than". See [9, pp. 105 ff].

holds if and only if $T_1(x, q(y_1, \dots, y_\nu))$ or $T_2(x, q(y_1, \dots, y_\nu))$, which, in turn, holds if and only if

$$\text{MayDo}\big(x, q(y_1, \dots, y_\nu)\big) \;\&\; \text{MayPass}\big(x, q(y_1, \dots, y_\nu)\big)$$

(see the list of one-agent formulas above).

If q is a descriptive condition, $T_i q$ $(1 \leqslant i \leqslant 7)$ is called a *basic np*-condition (*np* for normative position). By an *np*-condition, simpliciter, we mean a basic *np*-condition or a Boolean compound of such conditions.

We now present a construction which incorporates normative positions within the formal *cis* framework. We do this by constructing what will be called an *np-cis* with regard to a *cis* of descriptive conditions.

Let $\mathcal{M} = \langle M, \wedge, ', R\rangle$ be a *cis* (where Q is the similarity relation corresponding to R) with a domain of descriptive conditions $q_1, q_2, \dots$. Furthermore, let $T_M = \{T_i q \mid q \in M - \{\bot, \top\},\ 1 \leqslant i \leqslant 7\}$ and let T_M^* be the closure of T_M under $\wedge, '$. Then $\mathcal{T} = \langle T_M^*, \wedge, '\rangle$ is a Boolean algebra, called a *Boolean np-algebra* with regard to $\mathcal{M}$. An *np-cis* is defined so as having a Boolean *np*-algebra as its reduct.

Definition 8. If $\mathcal{T} = \langle T_M^*, \wedge, '\rangle$ is a Boolean *np*-algebra with regard to $\mathcal{M}$, then a *cis* $\mathcal{N} = \langle T_M^*, \wedge, ', R_N\rangle$ is an `np-cis` with regard to $\mathcal{M}$ if for any $q, r \in M$ it holds that

(1) if $i \neq j$, then $T_i q \wedge T_j q \; R_N \; \bot$ (for $i, j \in \{1, \dots, 7\}$),
(2) $\top \; R_N \; (T_1 q \vee \dots \vee T_7 q)$,
(3) $T_1 q \, Q_N \, T_1 q'$, $T_3 q \, Q_N \, T_3 q'$, $T_6 q \, Q_N \, T_6 q'$, $T_2 q \, Q_N \, T_4 q'$, $T_5 q \, Q_N \, T_7 q'$, and
(4) if $q Q r$, then $T_i q \, Q_N \, T_i r$.

Requirements (1)–(4) in the definition express restrictions on the relation R_N in an *np-cis* $\mathcal{N}$ and correspond to three features of one agent types in the Kanger–Lindahl theory. Thus requirement (1) expresses that $T_1 q, \dots, T_7 q$ are mutually incompatible, (2) that they are jointly exhaustive, and (3) that T_1, T_3, T_6 are neutral, while T_4 is the converse of T_2 and T_7 the converse of T_2. Requirement (4), finally, corresponds to the "extensionality" feature for combinations of operators Shall and Do in the Kanger–Lindahl theory.

As mentioned, $\mathcal{T}$ is the reduct of $\mathcal{N}$. Consequently, if a is an atom in $\mathcal{T}$ and not $a \, Q_N \bot$, then a is an atom in $\mathcal{N}$. Furthermore, if $\mathcal{N}$ is finite and a is an atom in $\mathcal{N}$, then there is an atom b in $\mathcal{T}$ such that $a \, Q_N \, b$ (see Lemmas 5 and 6). Finally, the Boolean relation $\leqslant_T$ of $\mathcal{T}$ is a subset of the relation R_N of $\mathcal{N}$.

To give a very simple application of Definition 8, consider the case of a pair $\mathcal{M}, \mathcal{T}$ where

$\mathcal{M} = \big\langle \{q_1, q_2, \bot, \top, \}, \wedge, ', R\big\rangle$ is a *cis* and q_1, q_2 are descriptive conditions,
$\mathcal{T} = \langle T_M^*, \wedge, '\rangle$ is a Boolean *np*-algebra with regard to $\mathcal{M}$.

Let us consider the problem which, up to Q_N-similarity, is the maximal number of atoms in an *np-cis* $\mathcal{N}$ with regard to $\mathcal{M}$. We will establish that, up to Q_N-similarity, there are at most 7 atoms (i.e., the quotient algebra of $\mathcal{N}$ has at most seven atoms). This can be seen as follows.

The set T_M with regard to $\mathcal{M}$ has the 14 elements $T_1q_1, \ldots, T_7q_1, T_1q_2, \ldots, T_7q_2$. In the *np*-algebra $\mathcal{T}$, let $+T_iq_j$ denote T_iq_j and let $-T_iq_j$ denote $(T_iq_j)'$. The atoms of $\mathcal{T}$ have the following form:

$$\left(\bigwedge_{i=1}^{7} \pm T_iq_1\right) \wedge \left(\bigwedge_{i=1}^{7} \pm T_iq_2\right).$$

The number of these atoms is $2^{7\cdot 2}$ (i.e., $2^{14} = 16\,384$). The fact that, up to Q_N-similarity, only seven atoms remain in an *np-cis* $\mathcal{N}$ with regard to $\mathcal{M}$, is due to the requirements of Definition 8.

We first establish the result of imposing requirement (1) of Definition 8 on the 2^{14} atoms of $\mathcal{T}$. Let us consider an atom a in $\mathcal{T} = \langle T_M^*, \wedge, '\rangle$ and let l denote the left conjunct and r the right conjunct of a. Each of l and r consists of seven conjuncts. First consider l. In order that $a = l \wedge r$ be an atom in $\mathcal{N}$, it is required that not $l\, R_N \bot$. Consequently, if the sign chosen for conjunct T_iq_1 of l is plus, the sign chosen for any conjunct T_jq_1 of l, where $i \neq j$, must be minus, since, according to requirement (1) in Definition 8, $T_iq_1 \wedge T_jq_1\, R_N \bot$ if $i \neq j$. Hence, for $a = l \wedge r$, if a is an atom in $\mathcal{N}$, there are only seven possible values for l:

$$\begin{aligned}
&T_1q_1 \wedge (T_2q_1)' \wedge \cdots \wedge (T_7q_1)',\\
&T_2q_1 \wedge (T_1q_1)' \wedge (T_3q_1)' \wedge \cdots \wedge (T_7q_1)',\\
&\vdots\\
&T_7q_1 \wedge (T_1q_1)' \wedge \cdots \wedge (T_6q_1)'.
\end{aligned}$$

If we consider r, analogously there are seven possible values for r with regard to q_2. Hence, due to requirement (1) of Definition 8, up to Q_N-similarity there can be at most 7^2 atoms in $\mathcal{N}$. Since, according to requirement (1) $T_iq_1 R_N (T_jq_1)'$ if $i \neq j$, these 49 Q_N-equivalence classes can be represented by the following members:

$$\text{(I)}\quad \begin{array}{llll}
T_1q_1 \wedge T_1q_2, & T_1q_1 \wedge T_2q_2, & \ldots & T_1q_1 \wedge T_7q_2\\
T_2q_1 \wedge T_1q_2, & T_2q_1 \wedge T_2q_2, & \ldots & T_2q_1 \wedge T_7q_2\\
\vdots & \vdots & \ldots & \vdots\\
T_7q_1 \wedge T_1q_2, & T_7q_1 \wedge T_2q_2, & \ldots & T_7q_1 \wedge T_7q_2.
\end{array}$$

List (I) is further reduced by imposing requirement (3) of Definition 8 together with requirement (1), taking into account that $q_2 = q_1'$. For instance, by (3) it holds that $T_4q_2 Q_N T_2q_1$. It follows, for example, that $T_3q_1 \wedge T_4q_2 Q_N T_3q_1 \wedge T_2q_1$, where only q_1 appears in the conjunction to the right. Accordingly, $T_3q_1 \wedge T_4q_2$ is no atom in $\mathcal{N}$, since (according to requirement 1) $T_3q_1 \wedge T_2q_1 R_N \bot$. It is easy to verify that, of the 49 classes represented by table (I), only 7 are such that their members are Q_N-different from $\bot$. This shows that, in the case we have in view, of the 49 Q_N-equivalences classes of atoms in $\mathcal{T}$, there are only at most 7 that contain Q_N-different atoms in $\mathcal{N}$. These classes can be represented by the following members selected from list (I):

$$(\text{II})\quad \begin{array}{l} T_1q_1 \wedge T_1q_2 \\ T_2q_1 \wedge T_4q_2 \\ T_3q_1 \wedge T_3q_2 \\ T_4q_1 \wedge T_2q_2 \\ T_5q_1 \wedge T_7q_2 \\ T_6q_1 \wedge T_6q_2 \\ T_7q_1 \wedge T_5q_2. \end{array}$$

The list (II) can be simplified. Since $q_2 = q_1'$, by requirement (3) of the definition each of the two conjuncts in any line of (II) is Q_N-related to the other conjunct and hence Q_N-related to the conjunction. Therefore, a further simplification is obtained by representing the 7 classes by the respective members,

$$(\text{III})\quad T_1q_1, \ldots, T_7q_1,$$

where only q_1 occurs. Thus any atom in $\mathcal{N}$ is Q_N-related to an item in list (III). This means that the 7 items in (III) represent the Q_N-equivalence classes of atoms that, at most, there can be in $\mathcal{N}$. Since, in our present application (where $\mathcal{M} = \langle\{q_1, q_2, \bot, \top\}, \wedge, '\rangle$ and so $q_2 = q_1'$), we are in fact concerned with the "one-condition case", this is what can be expected.

The example dealt with shows how Definition 8 works in the most simplistic case. The justification of the definition is, of course that it opens up for generalizations to many-conditions cases in a straightforward manner.

4.1.3. Liberty conditions

For seeing more clearly what various conditions in an *np-cis* amount to in deontic terms, the notion of *liberty conditions* can be introduced (cf. [9, pp. 106ff]). This device is available since each *np*-condition equals a Boolean compound of liberty conditions.

Since the elements of an *np-cis* are *np*-conditions, we choose to define liberty conditions in terms of disjunctions of basic *np*-conditions.

Definition 9. L_1, L_2, L_3 are operators on conditions such that, if q is a condition:

(1) L_1q is defined as: $T_1q \vee T_2q \vee T_3q \vee T_5q$.
(2) L_2q is defined as: $T_1q \vee T_2q \vee T_4q \vee T_6q$.
(3) L_3q is defined as: $T_1q \vee T_3q \vee T_4q \vee T_7q$.

Accordingly, it holds that,

$$\begin{array}{l} T_1q\,Q\,L_1q \wedge L_2q \wedge L_3q, \\ T_2q\,Q\,L_1q \wedge L_2q \wedge (L_3q)', \\ T_3q\,Q\,L_1q \wedge (L_2q)' \wedge L_3q, \\ T_4q\,Q\,(L_1q)' \wedge L_2q \wedge L_3q, \end{array}$$

$$T_5 q\, Q L_1 q \wedge (L_2 q)' \wedge (L_3 q)',$$
$$T_6 q\, Q (L_1 q)' \wedge L_2 q \wedge (L_3 q)',$$
$$T_7 q\, Q (L_1 q)' \wedge (L_2 q)' \wedge L_3 q.$$

Informally, the three liberty operators L_1, L_2 and L_3 can be called action permissibility, passivity permissibility and counter-action permissibility, respectively. In terms of May and Do we can read non-negated liberty conditions as follows.

Action permissibility: L_1

$$L_1 q(x_1, \ldots, x_\nu, x_{\nu+1}) \quad \text{iff} \quad \text{May Do}\big(x_{\nu+1}, q(x_1, \ldots, x_\nu)\big).$$

Passivity permissibility: L_2

$$L_2 q(x_1, \ldots, x_\nu, x_{\nu+1}) \quad \text{iff} \quad \text{May Pass}\big(x_{\nu+1}, q(x_1, \ldots, x_\nu)\big).$$

Counter-action permissibility: L_3

$$L_3 q(x_1, \ldots, x_\nu, x_{\nu+1}) \quad \text{iff} \quad \text{May Do}\big(x_{\nu+1}, q(x_1, \ldots, x_\nu)'\big).$$

These readings may prove to be helpful subsequently when dealing with examples of normative correlation in a normative system, where an *np-cis* of normative consequences is joined to a descriptive *cis* of grounds.

5. Normative systems

5.1. Fragments, joinings and connections

5.1.1. Basic definitions

Treating a normative system as a *cis* of conditions provides a convenient way of dealing with normative correlations in a normative system. If a_1 is a descriptive and a_2 is a normative condition, we will say that $a_1 R a_2$ describes a normative correlation for $\mathcal{S}$. For example, if a_1 is the (descriptive) condition of being less than fifteen years old and a_2 is the (normative) condition of being liable to punishment, the statement that from $\mathcal{S}$ it follows that a implies a_2', is represented by $a_1\ R\ a_2'$. In the *cis* model of the *Bqo* theory we are interested in, normative correlations will be studied primarily in terms of how a descriptive *cis* $\mathcal{B}_1$ and an *np-cis* $\mathcal{B}_2$ can be combined.

For the purpose of this study, in the present section a number of concepts will be introduced which will serve as tools for the analysis, together with lemmas or theorems relating to these concepts. Most of these concepts, lemmas, and theorems are of a general character in the sense that they belong to the level of the general *Bqo* theory rather than to the level of *cis* models. In the course of the exposition, however, the use of the concepts at the more specific *cis* level will be explained, with a view to normative systems.

Since, for the study of normative correlations, we need to distinguish between various parts of a *Bqo*, an important notion is that of a fragment $\mathcal{B}_i$ of a *Bqo* $\mathcal{B}$.

Definition 10. If $\mathcal{B} = \langle B, \wedge, ', R\rangle$ is a Boolean quasi-ordering, and $\langle B_i, \wedge, '\rangle$ is a subalgebra of $\langle B, \wedge, '\rangle$, and $R_i = R/B_i$, then the structure $\mathcal{B}_i = \langle B_i, \wedge, ', R_i\rangle$ will be called a `fragment` of $\mathcal{B}$.[27]

We note that if $\mathcal{B}$ is a *Bqo*, and $\mathcal{B}_i$ is a fragment of $\mathcal{B}$, then $\mathcal{B}_i$ is a *Bqo*.

The kinds of combinations of fragments that are in focus are those we call "joinings", "connections", and "couplings". It is appropriate to define these notions at the general *Bqo* level.

Definition 11. Let $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2$ be *Bqo*'s such that $\mathcal{B}_1$ and $\mathcal{B}_2$ are fragments of $\mathcal{B}$. A `joining` from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ is a pair $\langle b_1, b_2\rangle$ in $\mathcal{B}$ such that $b_1 \in B_1$, $b_2 \in B_2$, $b_1 R b_2$, not $b_1 R \bot$ and not $\top R b_2$. A joining $\langle b_1, b_2\rangle$ from $\mathcal{B}_1$ to $\mathcal{B}_2$ is called strict if $b_1 S b_2$.

The set of joinings from $\mathcal{B}_1, \mathcal{B}_2$ (in $\mathcal{B}$) will be denoted *Joining*$(\mathcal{B}_1, \mathcal{B}_2)$. In the context of a set *Joining*$(\mathcal{B}_1, \mathcal{B}_2)$, we will often speak of $\mathcal{B}_1$ as the "lower" fragment and of $\mathcal{B}_2$ as the "upper" fragment. In the same vein, if $\langle b_1, b_2\rangle \in$ *Joining*$(\mathcal{B}_1, \mathcal{B}_2)$, we will sometimes refer to b_1 as the "bottom" of the joining and of b_2 as the "top".

Thus, if $\mathcal{B}$ is a *cis* representing a normative system, where $\mathcal{B}_1$ is descriptive and $\mathcal{B}_2$ is normative, then *Joining*$(\mathcal{B}_1, \mathcal{B}_2)$ is a set of normative correlations.

An important concept in the analysis of normative systems is that of "completeness". There are different kinds of completeness, among which is "joining-completeness". At the *Bqo* level we say that the pair $\langle \mathcal{B}_1, \mathcal{B}_2\rangle$ is *upwards joining-complete* if for all $b_1 \in B_1$, where not $\top R b_1$, there is $b_2 \in B_2$, with not $\top R b_2$, such that $b_1 R b_2$. Conversely, $\langle \mathcal{B}_1, \mathcal{B}_2\rangle$ is *downwards joining-complete* if for all $b_2 \in B_2$, where not $b_2 R \bot$, there is $b_1 \in B_1$, with not $b_1 R \bot$, such that $b_1 R b_2$.

If $\mathcal{B}$ is a *cis* representing a normative system, and $\mathcal{B}_1$ is descriptive while $\mathcal{B}_2$ is normative, upwards joining-completeness means that to any descriptive condition in $\mathcal{B}_1$ there is correlated a normative condition, or "solution" in $\mathcal{B}_2$. Downwards joining-completeness means that any normative condition in $\mathcal{B}_2$ is correlated to a descriptive condition in $\mathcal{B}_1$.

The notion of "connections" is of particular interest to our inquiry. If $\mathcal{B}_1$ has a domain of descriptive conditions while the domain of $\mathcal{B}_2$ is normative, the set of connections from $\mathcal{B}_1$ to $\mathcal{B}_2$ can be thought of as representing the specific legal content, for example the result of legal enactment.[28] Thus, an important component in creating a normative system is the construction of several sets of ordered pairs of elements intended to be sets of connections, and modifications of the system implies a change of one or several sets of connections.

[27] The expression R/B_1 denotes the restriction of the relation R to the set B_1. We remind the reader of the definition of a subalgebra of a Boolean algebra. If $\langle B, \wedge, '\rangle$ is a Boolean algebra and A is a non-empty subset of B such that A is closed under the operations $\wedge$ and $'$, then $\langle A, \wedge_A, '_A\rangle$ is a subalgebra of $\langle B, \wedge, '\rangle$ where $\wedge_A$ and $'_A$ are restrictions of the operations $\wedge$ and $'$ to A. (That A is closed under the operations $\wedge$ and $'$ means that if $a, b \in A$ then $a \wedge_A b \in A$ and $a'_A \in A$.) If $\langle A, \wedge_A, '_A\rangle$ is a subalgebra of $\langle B, \wedge, '\rangle$ we often omit the subscript A and denote it simply $\langle A, \wedge, '\rangle$. Suppose that $\langle A, \wedge, '\rangle$ is a subalgebra of $\langle B, \wedge, '\rangle$ and let $\leqslant$ be the partial ordering determined by $\langle B, \wedge, '\rangle$ and $\leqslant_A$ the partial ordering determined by $\langle A, \wedge, '\rangle$. Then $\leqslant_A = \leqslant /A$ and $\inf \leqslant_A \{a, b\} = \inf \leqslant \{a, b\}$ and $\sup \leqslant_A \{a, b\} = \sup \leqslant \{a, b\}$.

[28] As will be discussed subsequently, this is especially plausible if each of $\mathcal{B}_1$ and $\mathcal{B}_2$ is conservatively reductible.

Definition 12. A `connection` from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ is a pair $\langle b_1, b_2\rangle$ such that the following four requirements are satisfied:

(i) $\langle b_1, b_2\rangle$ is a joining from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$.
(ii) There is $a_1 \in B_1 \backslash B_2$ and $a_2 \in B_2 \backslash B_1$ such that $a_1 R b_1$ and $b_2 R a_2$.
(iii) If $a_1 \in B_1$ and $b_1 R a_1 R b_2$ then $a_1 R b_1$.
(iv) If $a_2 \in B_2$ and $b_1 R a_2 R b_2$ then $b_2 R a_2$.

Requirements (iii)–(iv) are called the proximity principles. Intuitively, if $\langle b_1, b_2\rangle$ is a connection, then there is no element in B_1 or B_2 which is strictly between b_1 and b_2.

The set of connections from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ will be denoted $Conn(\mathcal{B}_1, \mathcal{B}_2)$.

We note that if $\langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, then $\langle b_2', b_1'\rangle \in Conn(\mathcal{B}_2, \mathcal{B}_1)$ (in $\mathcal{B}$). We call $\langle b_2', b_1'\rangle$ the *converse* of the connection $\langle b_1, b_2\rangle$.

5.1.2. Development of connection theory

Informally, the first theorem to be stated below says that in a finite *Bqo*, each joining such that $b_1 \in B_1 \backslash B_2$ and $b_2 \in B_2 \backslash B_1$ encompasses a connection.[29] Thus in a finite *cis* representing a normative system, if a descriptive and a normative condition are correlated there is always a closest correlation serving as a vehicle between them.

Theorem 13. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are fragments of a finite Bqo $\mathcal{B}$, and $\langle a_1, a_2\rangle$ is a joining from $\mathcal{B}_1$ to $\mathcal{B}_2$ such that $a_1 \in B_1 \backslash B_2, a_2 \in B_2 \backslash B_1$, then there is a connection $\langle b_1, b_2\rangle$ from $\mathcal{B}_1$ to $\mathcal{B}_2$ such that $a_1 R b_1$ and $b_2 R a_2$.*

Proof. We recall Definition 7. (1) Let $b_1 \in \mathfrak{lub}_{R_1}\{c \in B_1 \mid cRa_2\}$ and hence $b_1 \in \mathfrak{lub}_R\{c \in B_1 \mid cRa_2\}$. Then, since $a_1 \in \{c \in B_1 \mid cRa_2\}$, $a_1 R b_1$. (We note that, since $\mathcal{B}$ is finite, $\mathfrak{lub}_{R_1}\{c \in B_1 \mid cRa_2\} \neq \emptyset$, and since $\mathcal{B}_1$ is a fragment of $\mathcal{B}$, $\mathfrak{lub}_{R_1}\{c \in B_1 \mid cRa_2\} \subseteq \mathfrak{lub}_R\{c \in B_1 \mid cRa_2\}$. Next, let $b_2 \in \mathfrak{glb}_{R_2}\{c \in B_2 \mid b_1 Rc\}$, and hence, $b_2 \in \mathfrak{glb}_R\{c \in B_2 \mid b_1 Rc\}$. Then, since $a_2 \in \{c \in B_2 \mid b_1 Rc\}$, $b_2 R a_2$. Again, we note that, since $\mathcal{B}$ is finite, $\mathfrak{glb}_{R_2}\{c \in B_2 \mid b_1 Rc\} \neq \emptyset$, and since $\mathcal{B}_1$, $\mathcal{B}_2$ are fragments of $\mathcal{B}$, $\mathfrak{glb}_{R_2}\{c \in B_2 \mid b_1 Rc\} \subseteq \mathfrak{glb}_R\{c \in B_2 \mid b_1 Rc\}$. Finally, since $b_1 \in \mathfrak{lb}_R\{c \in B_2 \mid b_1 Rc\}$ and $b_2 \in \mathfrak{glb}_R\{c \in B_2 \mid b_1 Rc\}$, $b_1 R b_2$. Thus $a_1 R b_1$, $b_1 R b_2$, $b_2 R a_2$. Therefore, since $\langle a_1, a_2\rangle$ is a joining, $\langle b_1, b_2\rangle$ is a joining as well, and requirement (i) for connections is fulfilled.

(2) Since $a_1 \in B_1 \backslash B_2, a_2 \in B_2 \backslash B_1$ it furthermore follows that requirement (ii) is fulfilled.

(3) Suppose that $c_1 \in B_1$ and $b_1 R c_1 R b_2$. From $c_1 R b_2$ and $b_2 R a_2$ it follows that $c_1 R a_2$. Since $c_1 R a_2$ and $b_1 \in \mathfrak{lub}_R\{c \in B_1 | cRa_2\}$, $c_1 R b_1$, which proves that $\langle b_1, b_2\rangle$ satisfies requirement (iii) of a connection.

(4) Suppose that $c_2 \in B_2$ and $b_1 R c_2 R b_2$. $b_1 R c_2$ together with $b_2 \in \mathfrak{glb}_R\{c \in B_2 \mid b_1 Rc\}$ implies that $b_2 R c_2$, which shows that $\langle b_1, b_2\rangle$ satisfies requirement (iv) of a connection. □

[29] A more general version of the theorem, not presupposing finiteness, is proved as Theorem 17 in [17].

Let $Conn(\mathcal{B}_1, \mathcal{B}_2)$ denote the set of connections from $\mathcal{B}_1$ to $\mathcal{B}_2$. The theorem below (Theorem 15) states another fundamental property of $Conn(\mathcal{B}_1, \mathcal{B}_2)$. First we state a lemma.

Lemma 14. *If $\langle a_1, a_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ and $b_1 \in B_1,\ b_2 \in B_2$, then*:

(1) *If $a_1 R b_2$, then $a_2 R b_2$.*
(2) *If $b_1 R a_2$, then $b_1 R a_1$.*

Proof. (1) From $\langle a_1, a_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ it follows $a_1 R a_2$. The conjunction of $a_1 R a_2$ and $a_1 R b_2$ implies $a_1 R(a_2 \wedge b_2)$. Since $a_1 R(a_2 \wedge b_2)$ and $\langle a_1, a_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, $a_2 R(a_2 \wedge b_2)$. Consequently, $a_2 R b_2$.

(2) From $\langle a_1, a_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ it follows $a_1 R a_2$. The conjunction of $a_1 R a_2$ and $b_1 R a_2$ implies $(a_1 \vee b_1) R a_2$. Since $(a_1 \vee b_1) R a_2$ and $\langle a_1, a_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, $(a_1 \vee b_1) R a_1$. Thus $b_1 R a_1$. □

Theorem 15. *If $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, then $a_1 R b_1$ if and only if $a_2 R b_2$.*

Proof. Firstly, if $\langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ and $a_1 R b_1$, by transitivity $a_1 R b_2$. If $\langle a_1, a_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ and $a_1 R b_2$, then, by Lemma 14(1), $a_2 R b_2$. Secondly, if $\langle a_1, a_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ and $a_2 R b_2$, by transitivity $a_1 R b_2$. If $\langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ and $a_1 R b_2$, then, by Lemma 14(2) (exchanging a for b and vice versa), $a_1 R b_1$. □

Thus, in a *cis* model of a normative system, where $\mathcal{B}_1, a_1, b_1$, are descriptive, $\mathcal{B}_2, a_2, b_2$ normative, and $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle$ are among the "closest" normative correlations, it holds that a_1 implies b_1 if and only if a_2 implies b_2.

If $\mathcal{B}$ is finite and $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, we can form new connections by using conjunction and disjunction. First we state a lemma and a corollary.

Lemma 16. *If $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ and $c_1 \in B_1, c_2 \in B_2$, then,*
(i) *$c_1 R(a_2 \wedge b_2)$ implies $c_1 R(a_1 \wedge b_1)$.*
(ii) *$(a_1 \vee b_1) R c_2$ implies $(a_2 \vee b_2) R c_2$.*

Proof. (i) If $c_1 R(a_2 \wedge b_2)$, $c_1 R a_2$ and $c_1 R b_2$. Hence, if $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, according to Lemma 14, (2), $c_1 R a_1$ and $c_1 R b_1$, consequently, $c_1 R(a_1 \wedge b_1)$.

(ii) If $(a_1 \vee b_1) R c_2$, $a_1 R c_2$ and $b_1 R c_2$. Hence, if $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, according to Lemma 14, (1), $a_2 R c_2$ and $b_2 R c_2$, consequently, $(a_2 \vee b_2) R c_2$. □

Corollary 17. *Let $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$. Then, if $c_1 \in B_1$ and $c_2 \in B_2$,*

(i) *$a_1 \wedge b_1 R c_1 R a_2 \wedge b_2$ implies $c_1 Q a_1 \wedge b_1$,*
(ii) *$a_1 \vee b_1 R c_2 R a_2 \vee b_2$ implies $a_2 \vee b_2 Q c_2$.*

The corollary follows immediately from the preceding lemma.

Theorem 18. *Let $\mathcal{B}$ be finite with $\mathcal{B}_1$, $\mathcal{B}_2$ fragments of $\mathcal{B}$ and with $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$. Then,*

(1) *If $\langle a_1 \wedge b_1, a_2 \wedge b_2\rangle$ is a joining from $\mathcal{B}_1$ to $\mathcal{B}_2$, then there is $c_2 \in B_2$ such that $\langle a_1 \wedge b_1, c_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, and $c_2 R(a_2 \wedge b_2)$.*

(2) *If $\langle a_1 \vee b_1, a_2 \vee b_2\rangle$ is a joining, then there is $c_1 \in B_1$ such that $\langle c_1, a_2 \vee b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, and $(a_1 \vee b_1)Rc_1$.*

Proof. (1) From Theorem 13 it follows that there is $\langle c_1, c_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ such that $a_1 \wedge b_1 R c_1 R c_2 R a_2 \wedge b_2$. Hence, from Corollary 17, (i), it follows that $a_1 \wedge b_1 Q c_1$. Consequently there is c_2 such that $\langle a_1 \wedge b_1, c_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ and $c_2 R a_2 \wedge b_2$. The proof of (2) is analogous. □

If $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ and $\langle a_1 \wedge b_1, a_2 \wedge b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, then, figuratively speaking, the "R-strength" of $a_1 \wedge b_1$ with regard to $a_2 \wedge b_2$ equals the joint "R-strength" of a_1, a_2. However, it can be the case that there is $c_2 \in B_2$ with $\langle a_1 \wedge b_1, c_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, where $c_2 S(a_2 \wedge b_2)$.[30] Then, the "R-strength" of $a_1 \wedge b_1$ exceeds the joint "R-strength" of a_1, a_2. In the latter case, $a_1 \wedge b_1$ might be called an "organic whole" with regard to R and $a_2 \wedge b_2$.

Lemma 19. *Let $\mathcal{B}$ be finite with $\mathcal{B}_1$, $\mathcal{B}_2$ fragments of $\mathcal{B}$ and with $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$. Then,*

(1) *If $\langle a_2 \wedge b_2\rangle \in at(\mathcal{B}_2)$ and not $a_1 \wedge b_1\ R\ \bot$, then $\langle a_1 \wedge b_1, a_2 \wedge b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$.*

(2) *If $\langle a_1 \vee b_1\rangle \in dual\,at(\mathcal{B}_1)$ and not $\top\, Ra_2 \vee b_2$, then, $\langle a_1 \vee b_1, a_2 \vee b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$.*

Proof. (1) Since $a_1 R a_2$ and $b_1 R b_2$, $a_1 \wedge b_1 R a_2 \wedge b_2$. Since $a_2 \wedge b_2 R a_2$ and not $\top R a_2$, not $\top R a_2 \wedge b_2$. Therefore, since, moreover, not $a_1 \wedge b_1 R \bot$, $\langle a_1 \wedge b_1, a_2 \wedge b_2\rangle \in Joining(\mathcal{B}_1, \mathcal{B}_2)$. Hence, by Theorem 18, (1), there is $c_2 \in B_2$ such that $\langle a_1 \wedge b_1, c_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, and $c_2 R_2 a_2 \wedge b_2$. If $a_1 \wedge b_1 R c_2$ and not $a_1 \wedge b_1 R \bot$, not $c_2 R \bot$. Finally, if $c_2 R a_2 \wedge b_2$ and not $c_2 R \bot$, since $\langle a_2 \wedge b_2\rangle \in at(\mathcal{B}_2)$, $c_2 Q a_2 \wedge b_2$. Consequently, $\langle a_1 \wedge b_1, a_2 \wedge b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$. The proof of (2) is analogous. □

The following two, more special, lemmas will be needed later on.

Lemma 20. *If aRb, cRd, $\top Ra \vee c$ and $b \wedge dR\bot$, then aQc' and bQd'.*

Proof. First, since aRb and cRd, $a \wedge cRb \wedge d$, and since $b \wedge dQ\bot$, by transitivity, $a \wedge cR\bot$, and thus aRc'. Furthermore, since $a \vee cQ\top$, $c'Ra$, and so, aQc'. Next, since aRb and cRd, $a \vee cRb \vee d$. From this and $\top R(a \vee c)$, by transitivity, $\top R(b \vee d)$, i.e., $d'Rb$. From $b \wedge dQ\bot$ it follows bRd'. Hence, bQd'. □

Lemma 21. *If aRb, cRd, and $b \wedge dR\bot$, then $a \wedge cR\bot$.*

[30] We recall that S is the strict relation corresponding to R.

The proof is obvious.

Atoms, dual atoms and connections. We note that if $\langle a_1, a_2\rangle \in Joining(\mathcal{B}_1, \mathcal{B}_2)$, and $a_2 \in at(\mathcal{B}_2)$ (see Definition 3), then there is $b_1 \in B_1$ such that $\langle b_1, a_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$.

Lemma 22. *If $a_1 \in dual\ at(\mathcal{B}_1)$, and $a_2 \in at(\mathcal{B}_2)$, and $a_1 R a_2$, then $\langle a_1, a_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$.*

The proof is obvious.

Lemma 23. *If $a_1 \in dual\ at(\mathcal{B}_1)$, and $a_2 \in at(\mathcal{B}_2)$, and $a_1 R a_2$, and $\langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, then* (1) *$a_1 Q b_1$ and $a_2 Q b_2$ or* (2) *$a_1' Q b_1$ and $a_2' Q b_2$.*

Proof. Since a_2 atom in $\mathcal{B}_2$, for every $b_2 \in B_2$, such that not $b_2 Q \bot$: (i) $a_2 R b_2$ or (ii) $a_2 \wedge b_2 Q \bot$. First suppose $a_2 R b_2$. Then, since, by Lemma 22, $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, by Theorem 15, $a_1 R b_1$. Therefore, since a_1 is a dual atom in $\mathcal{B}_1, b_1 Q a_1$ and, hence, by Lemma 15, $a_2 Q b_2$. Next suppose $a_2 \wedge b_2 Q \bot$. Then, by Lemma 21, $a_1 \wedge b_1 Q \bot$, and so, not $b_1 R a_1$. Since a_1 is a dual atom in $\mathcal{B}_1$, $b_1 R a_1$ or $b_1' R a_1$.[31] Consequently, since not $b_1 R a_1$, $b_1' R a_1$, i.e., $a_1 \vee b_1 Q \top$. Therefore, in this case, by Lemma 20, $a_1' Q b_1$ and $a_2' Q b_2$. □

Corollary 24. *If $a_1 \in dual\ at(\mathcal{B}_1)$, and $a_2 \in at(\mathcal{B}_2)$, and $a_1 R a_2$, then,*

(1) *$\langle a_1, a_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$ and,*
(2) *if $\langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, then,*
  (i) *$a_1 Q b_1$ and $a_2 Q b_2$ or,*
  (ii) *$a_1' Q b_1$ and $a_2' Q b_2$.*

The corollary follows immediately from Lemmas 22 and 23.

5.2. Couplings and pair couplings

Definition 25. Suppose that $\mathcal{B}$ is a Boolean quasi-ordering and $\mathcal{B}_1$ and $\mathcal{B}_2$ are fragments of $\mathcal{B}$. Then $\langle b_1, b_2\rangle$ is a `coupling` from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ if the following three requirements are satisfied:

(i) $\langle b_1, b_2\rangle$ is a joining from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$.
(ii) There is $a_1 \in B_1 \backslash B_2$ and $a_2 \in B_2 \backslash B_1$ such that $a_1 R b_1$ and $b_2 R a_2$.
(iii) If $a_1 \in B_1, a_2 \in B_2$ and $a_1 R a_2$, then $a_1 R b_1$ and $b_2 R a_2$.

Let $Coupl(\mathcal{B}_1, \mathcal{B}_2)$ denote the set of couplings from $\mathcal{B}_1$ to $\mathcal{B}_2$. If $\langle b_1, b_2\rangle \in Coupl(\mathcal{B}_1, \mathcal{B}_2)$ then $\langle b_2', b_1'\rangle \in Coupl(\mathcal{B}_2, \mathcal{B}_1)$ and is called the converse of $\langle b_1, b_2\rangle$.

It is easy to see that every coupling from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ is also a connection. Furthermore, if there are more couplings than one from $\mathcal{B}_1$ to $\mathcal{B}_2$, these couplings are “similar”.

[31] We note that, since $a_1 R a_1 \vee b_1$, if a us a dual atom in $\mathcal{B}$, $\top\ R a_1 \vee b_1$ or $a_1 \vee b_1 Q a_1$. I.e., $b_1' R a_1$ or $b_1 R a_1$.

Theorem 26. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are fragments of a* Bqo *$\mathcal{B}$, and $\langle a_1, a_2\rangle \in Coupl(\mathcal{B}_1, \mathcal{B}_2)$ and $\langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, then $a_1 Q b_1$ and $a_2 Q b_2$.*

Proof. Since $b_1 R b_2$ and $\langle a_1, a_2\rangle \in Coupl(\mathcal{B}_1, \mathcal{B}_2)$, $b_1 R a_1$ and $a_2 R b_2$. Since $a_1 R a_2$ we get $b_1 R a_1 R b_2$ and $b_1 R a_2 R b_2$. Since $\langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, from requirement (iii) and (iv) of a connection follows $a_1 Q b_1$ and $a_2 Q b_2$. □

Definition 27. The set $\{\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\}$ is a *pair coupling* from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}$ if

(1) $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$,
(2) not $a_1 Q b_1$,
(3) for all $c_1 \in B_1$ and $c_2 \in B_2$ it holds that if $c_1 R c_2$, then either
  (i) $c_1 R a_1$ and $a_2 R c_2$, or,
  (ii) $c_1 R b_1$ and $b_2 R c_2$.

If $\{\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\}$ is a pair coupling from $\mathcal{B}_1$ to $\mathcal{B}_2$, then we say that the *converse pair coupling* from $\mathcal{B}_2$ to $\mathcal{B}_1$ is $\{\langle a_2', a_1'\rangle, \langle b_2', b_1'\rangle\}$.

Definition 28. Let $Sim(\langle a_1, a_2\rangle) = \{\langle b_1, b_2\rangle \mid a_1 Q b_1 \ \&\ a_2 Q b_2\}$.

Lemma 29. *$\{\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\}$ is a pair coupling iff $Conn(\mathcal{B}_1, \mathcal{B}_2) = Sim(\langle a_1, a_2\rangle) \cup Sim(\langle b_1, b_2\rangle)$.*

The proof is obvious.

The lemma says that if $\{\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\}$ is a pair coupling and $\langle c_1, c_2\rangle$ is a connection, then either $c_1 Q a_1$ and $c_2 Q a_2$, or $c_1 Q b_1$ and $c_2 Q b_2$. Thus, "up to similarity", there are only the two connections $\langle a_1, a_2\rangle, \langle b_1, b_2\rangle$.

Lemma 30. *If $Conn(\mathcal{B}_1, \mathcal{B}_2) = Sim(\langle a_1, a_2\rangle) \cup Sim(\langle b_1, b_2\rangle)$, then* (1) *$a_1 \wedge b_1 Q \perp$, or* (2) *$a_1 R b_1$ and $a_2 R b_2$ or* (3) *$b_1 R a_1$ and $b_2 R a_2$.*

Proof. Suppose that not $a_1 \wedge b_1 Q \perp$. Hence, $\langle a_1 \wedge b_1, a_2 \wedge b_2\rangle \in Joining(\mathcal{B}_1, \mathcal{B}_2)$. Consequently, by Theorem 18, there is a connection $\langle a_1 \wedge b_1, c_2\rangle$ such that $c_2 R a_2 \wedge b_2$. We distinguish between two cases. Suppose (i) that $\langle a_1 \wedge b_1, c_2\rangle \in Sim(\langle a_1, a_2\rangle)$, i.e., suppose that $c_2 Q a_2$ and $a_1 \wedge b_1 Q a_1$. Then $a_1 R b_1$ and, by Lemma 15, $a_2 R b_2$. By analogous reasoning, if $\langle a_1 \wedge b_1, c_2\rangle \in Sim(\langle b_1, b_2\rangle)$, then $b_1 R a_1$ and $b_2 R a_2$. □

Corollary 31. *If $Conn(\mathcal{B}_1, \mathcal{B}_2) = Sim(\langle a_1, a_2\rangle) \cup Sim(\langle b_1, b_2\rangle)$, and a_1, b_1 are R-unrelated, then $a_1 \wedge b_1 Q \perp$.*[32]

The proof is obvious.

[32] In the corollary, by "a_1, b_1 are R-unrelated" we mean that neither $a_1 R b_1$ nor $b_1 R a_1$.

5.2.1. Couplings dual atom to atom

Theorem 32. *If a_1 is a dual atom in $\mathcal{B}_1$, and a_2 an atom in $\mathcal{B}_2$, and $a_1 R a_2$, then, either $\langle a_1, a_2\rangle$ is a coupling, or, $\{\langle a_1, a_2\rangle, \langle a_1', a_2'\rangle\}$ is a pair coupling.*

Proof. The theorem follows from Corollary 24 and the Definitions 27 and 28 just given. □

Corollary 33. *If a_1 is a dual atom in $\mathcal{B}_1$, and a_2 an atom in $\mathcal{B}_2$, and $a_1 R a_2$, then,*

(1) $\{\langle a_1, a_2\rangle, \langle a_1', a_2'\rangle\}$ *is a pair coupling from $\mathcal{B}_1$ to $\mathcal{B}_2$ iff $a_1 Q a_2$.*

(2) $\langle a_1, a_2\rangle$ *is a coupling from $\mathcal{B}_1$ to $\mathcal{B}_2$ iff $a_1 S a_2$.*

Proof. (1) If $a_1 Q a_2$, then $a_1' Q a_2'$ and so, $\{a_1, a_2\rangle, \langle a_1', a_2'\rangle\}$ is a pair coupling. If $\{\langle a_1, a_2\rangle, \langle a_1', a_2'\rangle\}$ is a pair coupling, then $a_1 R a_2, a_1' R a_2'$ (i.e., $a_2 R a_1$), hence $a_1 Q a_2$. (2) If $\langle a_1, a_2\rangle$ is a coupling from $\mathcal{B}_1$ to $\mathcal{B}_2$, then $a_1 R a_2$, and, by (1), not $a_1 Q a_2$. □

6. Normative position fragments joined to descriptive fragments

6.1. An example

As appears from the foregoing, in this paper we represent a normative system by a *cis* with two fragments one of which is descriptive and the other is an *np-cis* (see above Section 4.1.2). In the present section we illustrate this representation by a simple example concerning the normative position of owners of real property in a legal system $\mathcal{S}$. We consider two fragments $\mathcal{B}_1 = \langle B_1, \wedge, ', R_1\rangle$ and $\mathcal{B}_2 = \langle B_2, \wedge, ', R_2\rangle$ of a *cis* $\mathcal{B}_0 = \langle B_0, \wedge, ', R_0\rangle$ representing $\mathcal{S}$. $\mathcal{B}_1$, called the "lower" fragment, is descriptive, while $\mathcal{B}_2$, called the "upper" fragment, is an *np-cis*. We recall that, since $\mathcal{B}_1, \mathcal{B}_2$ are fragments of $\mathcal{B}_0$, R_1 is the restriction of R_0 to B_1 and R_2 the restriction of R_0 to B_2. (Obviously, from this it follows that R_1, R_2 are subsets of R_0.) The sets *Joining*$(\mathcal{B}_1, \mathcal{B}_2)$ and *Conn*$(\mathcal{B}_1, \mathcal{B}_2)$ refer to normative correlations in $\mathcal{B}_0$ by the relation R_0, i.e., correlations between elements of B_1, called "grounds", and elements of B_2 called "consequences".

6.1.1. The two cis fragments considered

The descriptive lower fragment $\mathcal{B}_1$. We assume that conditions a_1 and a_2, appearing in the descriptive lower fragment $\mathcal{B}_1$ are as follows:

a_1. Being the owner of an estate E.[33]
b_1. Being the owner of an estate adjacent to estate E.

[33] Letter E is to be regarded as a parameter, in the sense of a quantity which is constant in a particular case considered, but which varies in different cases.

We furthermore assume that $\mathcal{B}_1$ is as depicted in the following diagram (where $\alpha \bowtie \beta$ is an abbreviation for $(\alpha \wedge \beta) \vee (\alpha' \wedge \beta')$).

$$
\begin{array}{lcccccccc}
\mathcal{B}_1 & & & & \top & & & & \\
 & & a_1 \vee b_1 & a_1 \vee b_1' & & a_1' \vee b_1 & a_1' \vee b_1' & & \\
 & a_1 & b_1 & a_1 \bowtie b_1 & & a_1 \bowtie b_1' & b_1' & a_1' & \\
 & & a_1 \wedge b_1 & a_1 \wedge b_1' & & a_1' \wedge b_1 & a_1' \wedge b_1' & & \\
 & & & & \bot & & & &
\end{array}
$$

We note that $\mathcal{B}_1$ coincides with its reduct and that, therefore, in $\mathcal{B}_1$, R_1 coincides with $\leqslant_1$. As appears from the diagram, it is assumed that conditions $a_1 \wedge b_1, a_1 \wedge b_1', a_1' \wedge b_1, a_1' \wedge b_1'$ are atoms in $\mathcal{B}_1$.

The upper normative fragment $\mathcal{B}_2$. Let conditions $q_1, \ldots, q_4$ be as follows:

q_1. Main building of estate *E* being painted white.
q_2. Main building on estate adjacent to *E* being painted white.
q_3. Cows of estate *E* entering land of adjacent estate.
q_4. Erecting a fence, going around estate *E* and adjacent estate.

Let $\mathcal{M} = \langle M, \wedge, {}' R\rangle$ be a *cis* such that the descriptive conditions q_1, q_2, q_3, q_4 are among the elements of its domain. Furthermore, as in Section 4.1.2, let $T_M = \{T_i q \mid q \in M - \{\bot, \top\}, 1 \leqslant i \leqslant 7\}$, let T_M^* be the closure of T_M under $\wedge, {}'$ and let $\mathcal{T} = \langle T_M^*, \wedge, {}'\rangle$ be a Boolean *np*-algebra with regard to $\mathcal{M}$. Finally, let $\mathcal{B}_2 = \langle T_M^*, \wedge, {}', R_2\rangle$ be an *np-cis* with regard to $\mathcal{M}$ (see above Definition 8). We recall that, since $\mathcal{T}$ is the reduct of $\mathcal{B}_2$, the Boolean relation $\leqslant_T$ of $\mathcal{T}$ is a subset of the relation R_2 of $\mathcal{B}_2$.

6.1.2. Joinings and connections

Joining assumptions. We assume that in the *cis* $\mathcal{B}_0$ representing legal system $\mathcal{S}$ the following holds:

(i) $a_1 \wedge b_1 R_0 T_1 q_1 \wedge T_1 q_2 \wedge T_1 q_3 \wedge T_1 q_4$,
(ii) $a_1 \wedge b_1' R_0 T_1 q_1 \wedge T_6 q_2 \wedge T_7 q_3 \wedge T_4 q_4$,
(iii) $a_1' \wedge b_1 R_0 T_6 q_1 \wedge T_1 q_2 \wedge T_4 q_3 \wedge T_4 q_4$,
(iv) $a_1' \wedge b_1' R_0 T_6 q_1 \wedge T_6 q_2 \wedge T_6 q_3 \wedge T_6 q_4$.

(Obviously, the conditions to the right of R_0 in (i)–(iv) are among the elements of T_M^*, i.e., the domain of $\mathcal{B}_2$.)

Given the intended interpretation of conditions $T_i q_j$ in terms of Shall, May and Do, the correlations (i)–(iv) are plausible for a legal system. This can be seen by inspection of the different grounds and consequences correlated. For this purpose, the notion of liberty conditions is useful (on liberty conditions, see above Section 4.1.3). To exemplify, $a_1 \wedge b_1$ means being the owner of both estate *E* and adjacent estate. This condition is a ground for $T_1 q_1 \wedge$ $T_1 q_2 \wedge$ $T_1 q_3 \wedge$ $T_1 q_4$, which is the *np*-condition denoting full freedom (operator T_1) with regard to all of $q_1, \ldots, q_4$ (painting the two buildings, letting the cows move around, erecting a surrounding fence). In contrast, $a_1 \wedge b_1'$ means owning estate E but not adjacent estate. This condition is ground for *np*-condition $T_1q_1 \wedge T_6q_2 \wedge T_7q_3 \wedge T_4q_4$.This *np*-condition denotes full freedom regarding the painting of building on estate E, no freedom to bring about or prevent painting of building on adjacent estate, obligation to see to it that cows from estate E do not enter land of adjacent estate, and, finally, freedom to prevent erection of the fence surrounding the estates and freedom to be passive about the matter, but no freedom to bring about the fence's being erected.

Connections from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}_0$. To illustrate connection theory (Section 5.1.2) by our example, let us assume that R_0 is such that each of (i)–(iv) expresses a connection from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}_0$. Thus, we assume that if

$$\begin{array}{lcl} \alpha_1 = a_1 \wedge b_1 & & \beta_1 = T_1q_1 \wedge T_1q_2 \wedge T_1q_3 \wedge T_1q_4, \\ \alpha_2 = a_1 \wedge b_1' & \text{and} & \beta_2 = T_1q_1 \wedge T_6q_2 \wedge T_7q_3 \wedge T_4q_4, \\ \alpha_3 = a_1' \wedge b_1 & & \beta_3 = T_6q_1 \wedge T_1q_2 \wedge T_4q_3 \wedge T_4q_4, \\ \alpha_4 = a_1' \wedge b_1' & & \beta_4 = T_6q_1 \wedge T_6q_2 \wedge T_6q_3 \wedge T_6q_4, \end{array}$$

then $\langle \alpha_i, \beta_i \rangle \in Conn(\mathcal{B}_1, \mathcal{B}_2)$, $(1 \leqslant i \leqslant 4)$. It can be verified that, given this assumption, it follows that each pair $\langle \alpha_j \vee \alpha_k, \beta_j \vee \beta_k \rangle$, $\langle \alpha_j \vee \alpha_k \vee \alpha_l, \beta_j \vee \beta_k \vee \beta_l \rangle$ belongs to $Conn(\mathcal{B}_1, \mathcal{B}_2)$ $(j, k, l \in \{1, \ldots, 4\})$. For a "two-disjuncts" pair $\langle \alpha_j \vee \alpha_k, \beta_j \vee \beta_k \rangle$ this is established by showing that neither in B_1 nor in B_2 is there any element, distinct from $\bot$, $\top$ strictly between $\alpha_j \vee \alpha_k$ and $\beta_j \vee \beta_k$. This is seen as follows. Firstly, since $\langle \alpha_j, \beta_j \rangle$, $\langle \alpha_k, \beta_k \rangle$ are connections, by Lemma 16, (ii), there is no $c_2 \in B_2$ such that $\alpha_j \vee \alpha_k R_0 c_2$ and $c_2 S_0 \beta_j \vee \beta_k$. Secondly, suppose there is $c_1 \in B_1$ such that $\alpha_j \vee \alpha_k S_0 c_1$ and $c_1 R_0 \beta_j \vee \beta_k$. Since R_0 restricted to B_1 is $\leqslant_1$, and α_j, α_k are atoms in $\mathcal{B}_1$, from $\alpha_j \vee \alpha_k S_0 c_1 R_0 \beta_j \vee \beta_k$ it follows that, among the remaining two atoms in $\mathcal{B}_1$, there is an atom α_m such that $\alpha_m S_0 c_1$, and hence such that $\alpha_m R_0 \beta_j \vee \beta_k$. This, however, is impossible since if α_m is an atom in $\mathcal{B}_1$, then, according to the assumptions above, $\langle \alpha_m, \beta_m \rangle$ is a connection for some β_m such that $\beta_m \wedge (\beta_j \vee \beta_k) Q_0 \bot$. The proof that a "three-disjuncts" pair $\langle \alpha_j \vee \alpha_k \vee \alpha_l, \beta_j \vee \beta_k \vee \beta_l \rangle$ belongs to $Conn(\mathcal{B}_1, \mathcal{B}_2)$ is analogous.[34]

The foregoing demonstration has a special point. An *np-cis* involving several embedded descriptive conditions $q_1, q_2, q_3, \ldots$ will have a very large number of elements and may seem difficult to handle as an upper normative fragment $\mathcal{B}_2$ of consequences correlated to a lower descriptive fragment $\mathcal{B}_1$ of grounds. However, when considering a small lower fragment $\mathcal{B}_1$ (representing the grounds that are of interest in a specific context), the set of connections from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}_0$ is small as well and can often be established without regard to the overwhelming number of remaining elements of $\mathcal{B}_2$. In this way, handling the *np-cis* construction is often easier than it might seem at first sight.

6.1.3. The expressive power of the np-cis construction

The theory of normative positions, as developed by Kanger, and by Lindahl in earlier works, does not integrate normative positions into the framework of a normative system.

[34] We note that the "four-disjuncts" pair $\langle \alpha_1 \vee \alpha_2 \vee \alpha_2 \vee \alpha_4, \beta_1 \vee \beta_2 \vee \beta_3 \vee \beta_4 \rangle$ is no connection from $\mathcal{B}_1$ to $\mathcal{B}_2$ in $\mathcal{B}_0$. (We have $\top R_1 \alpha_1 \vee \alpha_2 \vee \alpha_2 \vee \alpha_4$ and $\alpha_1 \vee \alpha_2 \vee \alpha_2 \vee \alpha_4 R_0 \beta_1 \vee \beta_2 \vee \beta_3 \vee \beta_4$, hence $\top R_0 \beta_1 \vee \beta_2 \vee \beta_3 \vee \beta_4$.) A pair $\langle \top, \top \rangle$ is not a joining or connection in the sense defined.

Moreover, and that is the point in focus now, this theory only deals with normative positions for one state of affairs F at a time. If normative positions for several states of affairs $F_1, \ldots, F_m$ are considered, they are dealt with separately for each F_i.[35] Obviously, the *np-cis* construction has much greater expressive power.

Let us illustrate the difference by a simple case. As shown above, to $\alpha_1 \vee \alpha_2$ (i.e., a_1, being owner of estate E) is connected the *np*-condition $\beta_1 \vee \beta_2$, which in the Boolean *np*-algebra $\mathcal{T}$, equals,

$$T_1q_1 \wedge \big((T_1q_2 \wedge T_1q_3 \wedge T_1q_4) \vee (T_6q_2 \wedge T_7q_3 \wedge T_4q_4)\big). \tag{δ}$$

In terms of the Boolean relation $\leqslant$ of $\mathcal{T}$, (δ) implies

$$T_1q_1 \wedge (T_1q_2 \vee T_6q_2) \wedge (T_1q_3 \vee T_7q_3) \wedge (T_1q_4 \vee T_4q_4) \tag{ε}$$

where the q_i are distributed over the four conjuncts. However, (ε) does not imply (δ) in terms of $\leqslant$ and neither does (ε) imply (δ) in terms of R_2 unless further special assumptions are made. In this sense, (δ) is richer in content than (ε). This difference in expressive power is important, since normative positions often go together in a “bundle” for a particular ground-condition, such as one regarding ownership. Condition a_1 encompasses two possibilities, i.e., $a_1 \wedge b_1$ (being the owner of both estates) and $a_1 \wedge b_1'$ (being the owner only of estate E). Condition (δ) exhibits the difference between, on one hand, the bundle (relating to q_2, q_3, q_4) that goes with $a_1 \wedge b_1$, namely the first of the two disjuncts in (δ), and on the other hand, the bundle that goes with $a_1 \wedge b_1'$, namely the second of these two disjuncts. This ”bundle character” of the normative consequence is not captured by (ε).

The analogue, in the present context, of the earlier Kanger–Lindahl construction dealing with one F at a time, would be a construction connecting (ε) rather than (δ) to a_1 as normative consequence, since in (ε) the q_i are distributed over the conjuncts and can be dealt with one at a time. As just shown, however, due to the ”bundle character” of the normative consequence, such a construction is less informative.

7. Conclusion

The theme of this paper has been the representation of normative systems within the algebraic theory of Boolean quasi-orderings, and the integration of normative positions within such a representation. In the course of the investigation a number of theoretical tools have been constructed for analysing various traits of normative systems. These tools divide mainly into three groups. One group encompasses the general theory of Boolean quasi-orderings and the models of this theory called condition implication structures (*cis*’s). Another group is that of normative positions and the framework of normative position *cis*’s (so-called *np-cis*’s). A third group, finally, is that of joinings, connections and couplings, aimed at the analysis of different kinds of correlations between descriptive and normative conditions.

[35] A notable exception is the work of Marek Sergot, see in particular [20].

The further development and full employment of the tools thus constructed is the task for a more comprehensive work. As regards putative next steps in the research program, however, some areas of interest can be hinted at. One area concerns the concepts of completeness and non-redundancy of a normative system or codex, an area closely related to the work of Alchourrón and Bulygin. Another area concerns changes of a normative system and the amplification of open systems. Still another problem area, dealt with specifically in some of our earlier papers, concerns legal concept formation and so-called intermediate concepts. Since, in our view, connections from descriptive to normative exhibit the kernel of a normative system, the concept of connection is an important tool for all three areas. Moreover, the distinction between various ways of connecting fragments of a normative system (atoms to atoms, atoms to dual atoms etc.) can serve as well for classification of different kinds of norms.

Further development of the theory of normative positions in the Kanger–Lindahl tradition has been accomplished by other scholars, in particular Sergot and Jones. A particular line of research concerns how the complex systems thus constructed can be incorporated within the framework for normative systems proposed by us. Part of this problem is the further development of the concept of an *np-cis*. Another part, however, is clarifying how the analysis of normative positions is related to the three problem areas concerning normative systems. As appears from the preceding sections, it is our belief that the representation of normative positions should be pursued as part of the representation of normative systems.

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[36] The paper, as well as our earlier joint papers, are the result of wholly joint work where the order of appearance of our author names has no significance.

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